Int. J. Solids Structures, 1970, Vol. 6, pp. 569 to 586. Pergamon Press. Printed in Great Britain

A DYNAMICAL THEORY OF TORSION

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Abstract—An approximate dynamical theory of torsion is developed which includes the effects of the warping and in-plane shearing deformations that, in general, accompany torsional deformation in cylindrical rods.

INTRODUCTION

EVER since Saint-Venant's famous memoir [1], the *static* torsion of isotropic elastic rods of arbitrary cross section has been one of the most widely known and most extensively analyzed classes of solutions within the framework of the exact, three-dimensional, linear theory of elasticity. On the other hand the only *dynamical* torsion problem which has yielded readily to analysis in the three-dimensional theory is the problem of torsional waves in a circular cylindrical rod which was formulated and solved by Pochhammer [2] (see also [3] and [4] for a discussion of this solution). For solving dynamical problems involving rods with other cross sectional shapes recourse has been made to either finding approximate solutions to the exact, three-dimensional equations (see, e.g. [5–10] and the review article [11]) or to constructing simplified, one-dimensional theories in which exact solutions of the approximate equations are obtained (see, e.g. [11–18] and [4, p. 429]).

Of the latter theories, the one most often employed in applications is also due to Saint-Venant [15]. It consists of the one-dimensional wave equation

$$\frac{\partial}{\partial z}\left(C\frac{\partial\theta}{\partial z}\right) = \rho J\frac{\partial^2\theta}{\partial t^2},$$

where θ is the angle of twist, z denotes the coordinate along the axis of the rod, t denotes time, ρ is the density of the material, J is the polar moment of inertia of the cross section and C (defined as the ratio of the applied torque to the resulting twist per unit length) is the static torsional rigidity calculated from the three-dimensional theory.

If C, ρ and J are constants the above equation predicts that torsional waves will propagate nondispersively with velocity $v_T = (C/\rho J)^{1/2}$. For a circular cylinder $C = \mu J$ where μ is the shear modulus, and thus the Saint-Venant theory predicts the velocity $v_T = (\mu/\rho)^{1/2}$ which is identical to the result calculated from the exact, Pochhammer solution. Although the Saint-Venant theory gives the correct result for a circular cylinder it is known experimentally that for other cross sectional shapes torsional waves are dispersive, especially at high frequencies [17, 19, 20].

One source of this dispersion is found in the interaction of the torsional motion with warping (axial) motion which is known to occur (at least in the static theory) for cross sections which are not circular but which is not accounted for explicitly in the dynamical

Saint-Venant theory†. Attempts have been made to incorporate the warping motion in a more direct manner into approximate theories by constructing theories which contain the effects of axial inertia or axial stress or both. A review of this literature can be found in the paper by Barr [17].

It has recently become clear that the coupling of the torsional with the warping motion is not the only source of dispersion of torsional waves. Kynch [21] (see also [7–11]) was apparently the first to notice that at high frequencies a distortion of the cross section in its own plane can occur. This motion (which we call contour–shear motion‡) can also couple with the torsional motion to produce dispersion. As we show in the sequel the frequency at which the contour–shear motion becomes significant is comparable to the frequency at which the warping motion becomes significant; thus an approximate theory which includes one of these motions should also include the other.

In the present paper an approximate dynamical theory of torsion is developed which includes the effects of both the warping and the contour–shear motions. The approximation is based on expansions of the displacements in terms of products of specified functions of the cross sectional coordinates and arbitrary functions of the axial coordinate and time together with a truncation procedure which retains only the torsional, contour–shear and warping motions. Analogous procedures have been employed by Mindlin and others (see, e.g. [22–27] and [16]) to arrive at hierarchies of approximate theories of plates and approximate theories for the extensional and flexural motions of rods. Prior attempts to construct approximate theories of torsion by this or similar procedures (see [12–18], [4, p. 429] and [11]) are all too restrictive to bring out the nature of the coupling between the torsional, contour–shear and warping motions.

1. EQUATIONS OF THE LINEAR THEORY OF ELASTICITY

We refer the equations to a rectangular cartesian coordinate system x_i, $i = 1, 2, 3$ and employ cartesian tensor notation, the summation convention for repeated indices, and the notations of a superposed dot indicating differentiation with respect to time, t, and a comma followed by an index indicating differentiation with respect to the corresponding spatial coordinate. With u_i, T_{ij}, S_{ij} and c_{ijkl} denoting, respectively, the components of the displacement, stress, strain and elastic stiffness tensors, with $\bar{t}_j(n_i)$ denoting the applied traction on the surface whose unit outward normal is n_i, and with ρ denoting the density, the equations of the linear theory of elasticity can be written in terms of

the variational equation of motion §:

$$\int_V [T_{ij,i} - \rho \ddot{u}_j]\delta u_j \, \mathrm{d}V + \int_A [\bar{t}_j(n_i) - n_i T_{ij}]\delta u_j \, \mathrm{d}A = 0, \tag{1}$$

where δu_j are arbitrary and independent infinitesimal variations in displacement and V is the volume enclosed by the surface A;

the strain–displacement relations:

$$S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}); \tag{2}$$

† The warping is taken into account indirectly in evaluating the torsional rigidity C.
‡ Kynch [21] classifies this motion as one of the family of screw modes.
§ For simplicity, we omit body forces.

and the constitutive relations:

$$T_{ij} = c_{ijkl}S_{kl}. \tag{3}$$

We note in passing that the variational equation of motion (1) contains not only stress equations of motion but also natural boundary conditions.

2. EXPANSION IN DOUBLE POWER SERIES

The cross section of the rod occupies a closed portion of the $x_1 - x_2$ plane bounded by the curve C; the axis of the rod is in the x_3 direction bounded by faces at $x_3 = \pm l$.

We expand the displacement components $u_j(x_1, x_2, x_3, t)$ in a series of powers of the coordinates x_1, x_2 in the cross section multiplying arbitrary functions of x_3 and t, viz.

$$u_j = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_1^n x_2^m u_j^{(n,m)}(x_3, t). \tag{4}$$

Inserting this expression into the variational equation of motion (1) and employing Green's Theorem to convert some of the surface integrals over the cross sectional area A_c into line integrals around the contour C, we find

$$\begin{aligned}
&\int_{-l}^{l} \mathrm{d}x_3 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Bigg\{\Bigg[T_{3j,3}^{(n,m)} - nT_{1j}^{(n-1,m)} - mT_{2j}^{(n,m-1)} + F_j^{(n,m)} \\
&\quad - \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \rho I_{n+pm+q} \ddot{u}_j^{(p,q)} \Bigg] \delta u_j^{(n,m)} \Bigg\} \\
&\quad + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \{[\bar{t}_j^{(n,m)}(n_3 = +1) - T_{3j}^{(n,m)}]\delta u_j^{(n,m)}\}_{x_3=+l} \\
&\quad + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \{[\bar{t}_j^{(n,m)}(n_3 = -1) - T_{3j}^{(n,m)}]\delta u_j^{(n,m)}\}_{x_3=-l} = 0,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
T_{ij}^{(n,m)} &\equiv \int_{A_c} x_1^n x_2^m T_{ij}\, \mathrm{d}A, \qquad I_{n+pm+q} \equiv \int_{A_c} x_1^{n+p} x_2^{m+q}\, \mathrm{d}A, \\
F_j^{(n,m)} &\equiv \oint_C \bar{t}_j(n_a) x_1^n x_2^m\, \mathrm{d}s, \qquad \bar{t}_j^{(n,m)}(n_3 = \pm 1) \equiv \int_{A_c(x_3=\pm l)} \bar{t}_j(n_3 = \pm 1) x_1^n x_2^m\, \mathrm{d}A.
\end{aligned} \tag{6}$$

In equations (5) and (6), $T_{ij}^{(n,m)}$ are stress-moments, I_{n+pm+q} are moments of the area of the cross section, $F_j^{(n,m)}$ are boundary forcing terms (arising from moments of the applied traction $\bar{t}_j(n_a)$ acting on the cylindrical surface with unit normal n_a, $a = 1, 2$), and $\bar{t}_j^{(n,m)}(n_3 = \pm 1)$ are, respectively, moments of the applied traction on the faces $x_3 = \pm l$. In the most general case, $T_{ij}^{(n,m)}$, I_{n+pm+q} and $F_j^{(n,m)}$ may be functions of x_3 and t and $\bar{t}_j^{(n,m)}(n_3 = \pm 1)$ functions of t.

The variations $\delta u_j^{(n,m)}$ in equation (5) are arbitrary and independent, from which follow the stress–moment equations of motion of order $n+m$,

$$T_{3j,3}^{(n,m)} - nT_{1j}^{(n-1,m)} - mT_{2j}^{(n,m-1)} + F_j^{(n,m)} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \rho I_{n+pm+q} \ddot{u}_j^{(p,q)}, \tag{7}$$

and the stress–moment boundary conditions of order $n+m$,

$$T_{3j}^{(n,m)} = \bar{t}_j^{(n,m)}(n_3) \quad \text{on} \quad x_3 = \pm l. \tag{8}$$

When the expansion (4) is substituted into the strain–displacement relations (2) and the terms are rearranged, we find that

$$S_{ij} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_1^n x_2^m S_{ij}^{(n,m)} \tag{9}$$

where

$$S_{ij}^{(n,m)} \equiv \tfrac{1}{2}[\delta_{j3}u_{i,3}^{(n,m)} + \delta_{i3}u_{j,3}^{(n,m)} + (n+1)(\delta_{1j}u_i^{(n+1,m)} + \delta_{1i}u_j^{(n+1,m)}) \\ + (m+1)(\delta_{2j}u_i^{(n,m+1)} + \delta_{2i}u_j^{(n,m+1)})], \tag{10}$$

and δ_{ij} is the Kronecker symbol. Multiplying the constitutive equation (3) by $x_1^n x_2^m$ and integrating over the cross section, the one-dimensional constitutive equations

$$T_{ij}^{(n,m)} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} I_{n+p\,m+q} c_{ijkl} S_{kl}^{(p,q)}, \tag{11}$$

are obtained. In the isotropic case (11) can be replaced by the simpler relation

$$T_{ij}^{(n,m)} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} I_{n+p\,m+q}[\lambda\delta_{ij}S_{kk}^{(p,q)} + 2\mu S_{ij}^{(p,q)}], \tag{12}$$

where λ and μ are the Lamé constants.

3. THE TORSIONAL EQUATIONS

Equations (7), (10) and (11) [or (12) in the isotropic case] comprise an infinite system of coupled one-dimensional equations for the variables $u_j^{(n,m)}$, and in the present form they are no easier to solve than the three-dimensional equations. It is expected however that a finite set of equations, obtained from the infinite set by an appropriate truncation procedure, can be found which will faithfully describe the dynamical behavior of a rod in a technologically important region of wavelengths and frequencies. For the formulation of approximate theories of plates and of circular rods by this method of expansion and truncation, exact solutions of the three-dimensional equations of elasticity (the Rayleigh–Lamb solution for plates and the Pochhammer–Chree solution for a circular rod) are available to aid in assessing the region of applicability of a particular truncation. For rods of arbitrary cross section, analogous solutions are not known and hence the truncation procedure relies more heavily on intuition. The final test of the theory, of course, is how well its predictions are confirmed by experiment.

In the present paper we limit our considerations to elastically isotropic rods which have two planes of geometrical symmetry at right angles to one another intersecting along the axis of the rod. The intersection of these planes with each cross sectional area is two perpendicular lines which we take as the x_1 and x_2 axes. With these assumptions the equations of Section 2 can be separated into four sets of equations which are uncoupled from one another. One set of equations governs the essentially "extensional" motions of the rod, i.e. $u_3^{(0,0)}$, the "fundamental" extensional motion, and higher order extensional motions such as radial shear motions, $u_2^{(0,1)}$ and $u_1^{(1,0)}$, axial shear motions, $u_3^{(0,2)}$ and $u_3^{(2,0)}$, etc. Two sets of equations govern the essentially "flexural" motions of the rod, i.e. the two "fundamental" flexural motions, $u_1^{(0,0)}$ and $u_2^{(0,0)}$, and their higher order flexural motions such as the axial shear motions, $u_3^{(1,0)}$ and $u_3^{(0,1)}$, etc. Equations analogous to those we would obtain here for the relatively low order extensional and flexural motions have already received a considerable amount of attention in the literature, hence we shall concentrate our efforts on the third set of equations, namely those governing what we shall call the *generalized torsional motions.*

Recalling that we utilize the sum $n+m$ to define the order of the various equations of motion (7), we find that there are no zeroth order equations governing generalized torsional motions. The first order, second order and third order equations which do govern torsional motions are†:

first order:

$$T_{31,3}^{(0,1)} - T_{12}^{(0,0)} = \rho[I_{02}\ddot{u}_1^{(0,1)} + I_{04}\ddot{u}_1^{(0,3)} + I_{22}\ddot{u}_1^{(2,1)} + \ldots], \tag{13}$$

$$T_{32,3}^{(1,0)} - T_{12}^{(0,0)} = \rho[I_{20}\ddot{u}_2^{(1,0)} + I_{40}\ddot{u}_2^{(3,0)} + I_{22}\ddot{u}_2^{(1,2)} + \ldots]; \tag{14}$$

second order:

$$T_{33,3}^{(1,1)} - T_{31}^{(0,1)} - T_{32}^{(1,0)} = \rho[I_{22}\ddot{u}_3^{(1,1)} + I_{24}\ddot{u}_3^{(1,3)} + I_{42}\ddot{u}_3^{(3,1)} + \ldots]; \tag{15}$$

third order:

$$T_{31,3}^{(0,3)} - 3T_{12}^{(0,2)} = \rho[I_{04}\ddot{u}_1^{(0,1)} + I_{06}\ddot{u}_1^{(0,3)} + I_{24}\ddot{u}_1^{(2,1)} + \ldots], \tag{16}$$

$$T_{32,3}^{(3,0)} - 3T_{12}^{(2,0)} = \rho[I_{40}\ddot{u}_2^{(1,0)} + I_{60}\ddot{u}_2^{(3,0)} + I_{42}\ddot{u}_2^{(1,2)} + \ldots], \tag{17}$$

$$T_{31,3}^{(2,1)} - 2T_{11}^{(1,1)} - T_{12}^{(2,0)} = \rho[I_{22}\ddot{u}_1^{(0,1)} + I_{24}\ddot{u}_1^{(0,3)} + I_{42}\ddot{u}_1^{(2,1)} + \ldots], \tag{18}$$

$$T_{32,3}^{(1,2)} - 2T_{22}^{(1,1)} - T_{12}^{(0,2)} = \rho[I_{22}\ddot{u}_2^{(1,0)} + I_{42}\ddot{u}_2^{(3,0)} + I_{24}\ddot{u}_2^{(1,2)} + \ldots], \tag{19}$$

where in each case we have written out only the first few inertia terms. Fourth and higher order equations are easily obtained from equation (7).

It is clear from an examination of equations (13), (14) and (15) and their associated constitutive equations that a theory truncated at what we might call, roughly, second order (in both the displacements and the equations of motion) would already introduce new phenomena in comparison to the classical theory of torsional motion which deals only with the special case $u_1^{(0,1)} = -u_2^{(1,0)}$. A theory which introduces no *a priori* relationship between the variables $u_1^{(0,1)}$, $u_2^{(1,0)}$, $u_3^{(1,1)}$ (see Fig. 1) will contain the phenomenon of shear

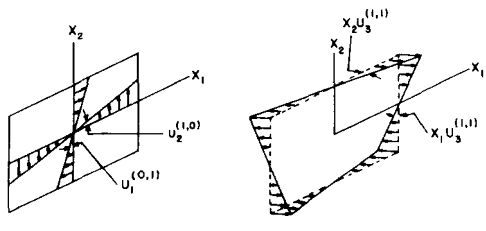

FIG. 1. Generalized torsional displacements of orders one and two.

† We omit body forces and the boundary forcing terms $F_j^{(n,m)}$ although they can be carried through the argument in a straightforward manner.

in the cross section ($u_1^{(0,1)} \neq -u_2^{(1,0)}$) and also an explicit description of the effects of the warping motion $u_3^{(1,1)}$, neither of which are included in the classical theory. Furthermore, such a theory should not be too unwieldy from the computational point of view. We thus seek a truncation which uncouples the first and second order equations governing $u_1^{(0,1)}$, $u_2^{(1,0)}$ and $u_3^{(1,1)}$ from all the higher order equations.

To begin the truncation, we neglect the effects of all components of generalized torsional displacement of order four and greater. This eliminates the need to consider stress–moment equations of order four and greater. To uncouple the first and second order equations from the third order equations we neglect the contribution of the inertia terms of order three, i.e. terms involving $\ddot{u}_1^{(0,3)}$, $\ddot{u}_1^{(2,1)}$, $\ddot{u}_2^{(3,0)}$, $\ddot{u}_2^{(1,2)}$, in the first order stress–moment equations. Furthermore, we eliminate the coupling to the third order displacements $u_1^{(2,1)}$ and $u_2^{(1,2)}$ (through the constitutive relations for $T_{11}^{(1,1)}$, $T_{22}^{(1,1)}$ and $T_{33}^{(1,1)}$) by setting $T_{11}^{(1,1)} = T_{22}^{(1,1)} = 0$ and allowing the strains $S_{11}^{(1,1)}$ and $S_{22}^{(1,1)}$ to occur freely. The displacements $u_1^{(2,1)}$ and $u_2^{(1,2)}$ can then be eliminated from the constitutive equation for $T_{33}^{(1,1)}$ (using the conditions $T_{11}^{(1,1)} = T_{22}^{(1,1)} = 0$ to solve for $u_1^{(2,1)}$ and $u_2^{(1,2)}$ in terms of $u_3^{(1,1)}$) with the result

$$T_{33}^{(1,1)} = I_{22}\left[\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\right]S_{33}^{(1,1)} = EI_{22}u_{3,3}^{(1,1)}, \tag{20}$$

where E is Young's Modulus.

The truncation is now logically completed, however we shall take it one step further by making certain adjustments analogous to those which have been employed so successfully by Mindlin and others in constructing approximate theories of plates and rods. In order to compensate in part for the restrictive assumptions that have been made about the functional dependence of the displacements on the coordinates in the cross section of the rod, it is desirable to introduce some free parameters (or "correction factors") into the approximate theory which can then be adjusted to match some feature or features of the solutions obtained from the approximate theory with those obtained from the three-dimensional theory or from experiments. Obviously, these parameters can be introduced in a variety of ways. Since the assumptions about the spatial dependence of the displacements gives rise, in the present theory, to the occurrence of the various moments of the area, a natural and convenient way to introduce correction factors is to replace each of the moments of area I_{00}, I_{02}, I_{20}, I_{22} which occur in this theory by a corrected moment, viz. I_{00}^* ($\equiv \kappa_{00}I_{00}$), I_{02}^* ($\equiv \kappa_{02}I_{02}$), I_{20}^* ($\equiv \kappa_{20}I_{20}$), I_{22}^* ($\equiv \kappa_{22}I_{22}$) in all of the equations. This scheme for introducing parameters has the advantage of leaving the symmetry of the equations unchanged. Methods for determining $\kappa_{00}, \kappa_{02}, \kappa_{20}$ and κ_{22} will be discussed further in Section 8.

The results of the forementioned truncation and adjustment procedure are the following equations for the determination of the quantities $u_1^{(0,1)}$, $u_2^{(1,0)}$ and $u_3^{(1,1)}$:

the stress–moment equations of motion:

$$\begin{aligned} T_{31,3}^{(0,1)} - T_{12}^{(0,0)} &= \rho I_{02}^*\ddot{u}_1^{(0,1)}, \\ T_{32,3}^{(1,0)} - T_{12}^{(0,0)} &= \rho I_{20}^*\ddot{u}_2^{(1,0)}, \\ T_{33,3}^{(1,1)} - T_{31}^{(0,1)} - T_{32}^{(1,0)} &= \rho I_{22}^*\ddot{u}_3^{(1,1)}; \end{aligned} \tag{21}$$

the constitutive equations:

$$\begin{aligned} T_{12}^{(0,0)} &= \mu I_{00}^*(u_2^{(1,0)}+u_1^{(0,1)}),\\ T_{31}^{(0,1)} &= \mu I_{02}^*(u_{1,3}^{(0,1)}+u_3^{(1,1)}),\\ T_{32}^{(1,0)} &= \mu I_{20}^*(u_{2,3}^{(1,0)}+u_3^{(1,1)}),\\ T_{33}^{(1,1)} &= EI_{22}^* u_{3,3}^{(1,1)}; \end{aligned} \tag{22}$$

and appropriate boundary conditions at the ends $x_3 = \pm l$ of the rod. For example at a "stress free" end the appropriate conditions are

$$T_{31}^{(0,1)} = T_{32}^{(1,0)} = T_{33}^{(1,1)} = 0. \tag{23}$$

Inserting (22) into (21) and assuming that μ, E, I_{02}^*, I_{20}^*, I_{22}^* are independent of x_3, the following

"*displacement*" *equations of motion:*

$$\begin{aligned} \mu I_{02}^*(u_{1,33}^{(0,1)}+u_{3,3}^{(1,1)})-\mu I_{00}^*(u_1^{(0,1)}+u_2^{(1,0)}) &= \rho I_{02}^*\ddot{u}_1^{(0,1)},\\ \mu I_{20}^*(u_{2,33}^{(1,0)}+u_{3,3}^{(1,1)})-\mu I_{00}^*(u_1^{(0,1)}+u_2^{(1,0)}) &= \rho I_{20}^*\ddot{u}_2^{(1,0)},\\ EI_{22}^* u_{3,33}^{(1,1)}-\mu I_{02}^*(u_{1,3}^{(0,1)}+u_3^{(1,1)})-\mu I_{20}^*(u_{2,3}^{(1,0)}+u_3^{(1,1)}) &= \rho I_{22}^*\ddot{u}_3^{(1,1)}, \end{aligned} \tag{24}$$

are obtained. In the sections that follow we shall discuss some of the features of these equations.

4. STATIC TORSION

In the static case a solution of equations (24) is

$$u_2^{(1,0)} = -u_1^{(0,1)} = x_3\frac{\theta_0}{l}, \qquad u_3^{(1,1)} = \frac{(I_{02}^*-I_{20}^*)}{(I_{02}^*+I_{20}^*)}\frac{\theta_0}{l}, \tag{25}$$

where θ_0/l is a constant twist per unit length which remains to be determined. Inserting (25) into (22) we see that $T_{12}^{(0,0)} = T_{33}^{(1,1)} = 0$ and

$$T_{32}^{(1,0)} = -T_{31}^{(0,1)} = \frac{2\mu I_{02}^* I_{20}^*}{I_{02}^*+I_{20}^*}\frac{\theta_0}{l}. \tag{26}$$

The *resultant torque* T acting at any cross section is given by the expression

$$T = T_{32}^{(1,0)}-T_{31}^{(0,1)} = \frac{4\mu I_{02}^* I_{20}^*}{I_{02}^*+I_{20}^*}\frac{\theta_0}{l}, \tag{27}$$

from which the torsional rigidity C defined by

$$C \equiv \frac{T}{(\theta_0/l)} = \frac{4\mu I_{02}^* I_{20}^*}{I_{02}^*+I_{20}^*} \tag{28}$$

is obtained. We note that the torsional rigidity (28) (with $\kappa_{02} = \kappa_{20} = 1$) is exact for an elliptical cross section [28], and, of course, for its limiting case the circle. It is also exact for an infinitely long, thin plate [29]. For other cross sectional shapes the formula (28) can be forced to give the exact value of the torsional rigidity by a suitable choice of the correction factors κ_{02}, κ_{20}. This procedure is discussed further in Section 8.

It is also worth noting that the solution (25) corresponds to the classical notion of torsion with unrestrained warping since $T_{33}^{(1,1)} = 0$ throughout the rod. If the warping displacement $u_3^{(1,1)}$ is restrained in any way that is incompatible with the assumptions (25), say by requiring it to vanish at $x_3 = \pm l$, then additional solutions of equations (24) are required in order to satisfy the end conditions. The same comments apply if the tractions at the end faces are distributed in some special way that is incompatible with (26) and/or the results $T_{12}^{(0,0)} = 0$, $T_{33}^{(1,1)} = 0$. Analogous situations have been treated to some extent in the three-dimensional theory where they are referred to as "end problems" (see, e.g. [30, 31]). In the three-dimensional theory it is found that solutions which decay exponentially from the ends are required in order to solve the end problems for a finite rod. In Section 9 we show that such solutions are contained in equations (24) (at least for the special case of a circular rod), and we believe that the present approximate theory may be very useful in treating these problems.

5. ALTERNATIVE FORM OF THE EQUATIONS

We introduce the definitions

$$J_+ \equiv I_{20}^* + I_{02}^*, \qquad J_- \equiv I_{20}^* - I_{02}^*, \qquad \delta \equiv J_-/J_+,$$
$$r^2 \equiv \frac{I_{02}^* + I_{20}^*}{I_{00}^*}, \qquad r_s^2 \equiv \frac{I_{02}^* I_{20}^*}{I_{00}^*(I_{02}^* + I_{20}^*)} = \frac{r^2(1-\delta^2)}{4}, \qquad r_w^2 \equiv \frac{I_{22}^*}{I_{02}^* + I_{20}^*}, \tag{29}$$

noting, in passing, that the six quantities J_+, J_-, δ, r, r_s and r_w are not all independent. Employing these definitions, equations (24) in the variables $u_1^{(0,1)}$, $u_2^{(1,0)}$, $u_3^{(1,1)}$, may be transformed to equations in the variables

$$\begin{aligned}
\psi_1 &\equiv \tfrac{1}{2}(u_2^{(1,0)} + u_1^{(0,1)}),\\
\psi_2 &\equiv \frac{1}{(1-\delta^2)}\left[\delta\left(\frac{u_2^{(1,0)} + u_1^{(0,1)}}{2}\right) + \left(\frac{u_2^{(1,0)} - u_1^{(0,1)}}{2}\right)\right],\\
\psi_3 &\equiv r_s u_3^{(1,1)},
\end{aligned} \tag{30}$$

by taking appropriate linear combinations of the first two of equations (24) and rewriting the third. The result is

$$\begin{aligned}
r_s^2\psi_{1,33} - \psi_1 + r_s\psi_{3,3} &= (\rho/\mu)r_s^2\ddot{\psi}_1,\\
(1-\delta^2)r_s^2\psi_{2,33} + \delta r_s\psi_{3,3} &= (1-\delta^2)(\rho/\mu)r_s^2\ddot{\psi}_2,\\
-(1-\delta^2)r_s\psi_{1,3} - \delta(1-\delta^2)r_s\psi_{2,3} + (E/\mu)r_w^2\psi_{3,33} - \psi_3 &= (\rho/\mu)r_w^2\ddot{\psi}_3,
\end{aligned} \tag{31}$$

where we note that ψ_2 is absent from the first equation and ψ_1 from the second equation. In terms of ψ_i, $i = 1, 2, 3$, we have

$$\begin{aligned}
u_1^{(0,1)} &= (1+\delta)\psi_1 - (1-\delta^2)\psi_2,\\
u_2^{(1,0)} &= (1-\delta)\psi_1 + (1-\delta^2)\psi_2,\\
u_3^{(1,1)} &= (1/r_s)\psi_3,
\end{aligned} \tag{32}$$

and

$$
\begin{aligned}
T_{12}^{(0,0)} &= \frac{2\mu J_+}{r^2}\psi_1, \\
T_{31}^{(0,1)} &= \frac{\mu J_+}{2r_s}(1-\delta)[(1+\delta)r_s\psi_{1,3}-(1-\delta^2)r_s\psi_{2,3}+\psi_3], \\
T_{32}^{(1,0)} &= \frac{\mu J_+}{2r_s}(1+\delta)[(1-\delta)r_s\psi_{1,3}+(1-\delta^2)r_s\psi_{2,3}+\psi_3], \\
T_{33}^{(1,1)} &= \frac{Er_w^2 J_+}{r_s^2}[r_s\psi_{3,3}].
\end{aligned}
\tag{33}
$$

The formulation of the torsional equations in terms of the variables ψ_i is equivalent to the formulation in terms of the variables $u_1^{(0,1)}$, $u_2^{(1,0)}$ and $u_3^{(1,1)}$; furthermore in some problems it is a more convenient formulation to work with because of the partial uncoupling which has been effected in equations (31).

6. VIBRATIONS OF AN INFINITE ROD AT ITS CUT-OFF FREQUENCIES

Consider solutions of equations (31) of the form

$$\psi_i = A_i e^{j\omega t}, \qquad i = 1, 2, 3, \tag{34}$$

where the A_i are constants and $j = (-1)^{\frac{1}{2}}$. These solutions correspond to limiting forms of waves in the rod as the wavelength becomes infinite. Equations (31) reduce to

$$
\begin{aligned}
A_1[(\rho/\mu)r_s^2\omega^2-1] &= 0, \\
A_2[(\rho/\mu)(1-\delta^2)r_s^2\omega^2] &= 0, \\
A_3[(\rho/\mu)r_w^2\omega^2-1] &= 0,
\end{aligned}
\tag{35}
$$

which for $\omega \neq 0$ admit the following two solutions:

(a) *Warping cut-off mode*

$$
\begin{aligned}
&A_1 = A_2 = 0, \qquad A_3 \neq 0, \\
&\omega_w^2 = \frac{\mu}{\rho}\frac{1}{r_w^2} = \frac{\mu}{\rho}\frac{I_{02}^*+I_{20}^*}{I_{22}^*},
\end{aligned}
\tag{36}
$$

(b) *Contour-shear cut-off mode*

$$
\begin{aligned}
&A_1 \neq 0, \qquad A_2 = A_3 = 0, \\
&\omega_s^2 = \frac{\mu}{\rho}\frac{1}{r_s^2} = \frac{\mu}{\rho}\frac{I_{00}^*(I_{02}^*+I_{20}^*)}{I_{02}^*I_{20}^*}.
\end{aligned}
\tag{37}
$$

The warping cut-off mode is a vibration in which $u_1^{(0,1)}$ and $u_2^{(1,0)}$ are both zero, i.e. there is no displacement in the plane of the cross section, and in which the warping displacement $u_3^{(1,1)}$ (see Fig. 1) varies periodically with time but has no dependence on the spatial coordinate along the rod. The contour–shear cut-off mode is a vibration in which $u_3^{(1,1)}$ is zero, i.e. there is no warping and in which $u_1^{(0,1)}$ and $u_2^{(1,0)}$ combine to produce a shear deformation (in the plane of the cross section) which varies periodically with time but has no dependence on the spatial coordinate along the rod.

The problem of determining cut-off modes and their associated frequencies can also be formulated and solved within the framework of the three-dimensional theory of elasticity, and as we show in Section 8 these solutions can be employed to establish the values of some of the correction factors.

7. PLANE WAVES IN AN INFINITE ROD

Consider solutions of equations (31) of the form

$$\begin{aligned} \psi_1 &= A_1 \sin(\xi x_3 - \omega t), \\ \psi_2 &= A_2 \sin(\xi x_3 - \omega t), \\ \psi_3 &= A_3 \cos(\xi x_3 - \omega t), \end{aligned} \tag{38}$$

where A_1, A_2 and A_3 are constants. Inserting (38) into (31) we obtain the secular equation

$$\begin{vmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = 0, \tag{39}$$

where

$$\begin{aligned} a_{11} &= \Omega^2 - 1 - \phi^2, \\ a_{22} &= (1-\delta^2)(\Omega^2 - \phi^2), \\ a_{33} &= (1-\delta^2)^{-1}[\Omega^2(r_w/r_s)^2 - 1 - 2(1+\nu)(r_w/r_s)^2\phi^2], \\ a_{13} &= -\phi, \\ a_{23} &= -\delta\phi, \end{aligned} \tag{40}$$

and

$$\phi \equiv \xi r_s, \qquad \Omega \equiv \omega/\omega_s, \qquad E/\mu = 2(1+\nu) \tag{41}$$

in which ν is Poisson's ratio. Equation (39) is an algebraic equation which is cubic in both Ω^2 and ϕ^2. Only real, positive values of Ω have physical significance however ϕ can be real, imaginary or complex, and for a given Ω if ϕ is a root so also is $-\phi$. For a given value of Ω there will be three values of ϕ^2 which satisfy (39), say ϕ_i^2, $i = 1, 2, 3$. For each of these values we obtain the amplitude ratios

$$\frac{A_1^i}{A_3^i} = \frac{\phi_i}{\Omega^2 - 1 - \phi_i^2}, \qquad \frac{A_2^i}{A_3^i} = \frac{\delta\phi_i}{(1-\delta^2)(\Omega^2 - \phi_i^2)}. \tag{42}$$

The details of the dispersion relation and the amplitude ratios depend of course upon the material properties of the rod and upon the geometry of the cross section however several general features are worth noting. At infinite wavelength ($\phi = 0$) the cut-off modes of Section 6 are recovered. For low frequencies and long wavelengths, i.e. $\Omega \ll 1$, $\phi \ll 1$, we find

$$\Omega^2 = (1-\delta^2)\phi^2 \tag{43}$$

from which, employing (29), (37) and (41) we obtain the asymptotic group (and phase) velocity

$$v_g = \left[\frac{d\omega}{d\xi}\right]_{\xi=0} = \left[\frac{\mu}{\rho}\frac{4I^*_{02}I^*_{20}}{(I^*_{02}+I^*_{20})^2}\right]^{\frac{1}{2}}. \tag{44}$$

This result is discussed further in the following section.

8. THE COEFFICIENTS $\kappa_{00}, \kappa_{02}, \kappa_{20}, \kappa_{22}$

In Section 3, four parameters $\kappa_{00}, \kappa_{02}, \kappa_{20}, \kappa_{22}$ were introduced into the equations of the generalized torsion theory by replacing the moments of the area $I_{00}, I_{02}, I_{20}, I_{22}$ by corrected moments I^*_{00} ($\equiv \kappa_{00}I_{00}$), I^*_{02} ($\equiv \kappa_{02}I_{02}$), I^*_{20} ($\equiv \kappa_{20}I_{20}$), I^*_{22} ($\equiv \kappa_{22}I_{22}$). The values of these four parameters can be obtained by matching solutions of the equations of the approximate theory with solutions of the equations of the exact, three-dimensional theory in four suitably chosen problems. In the present paper we follow the program of matching solutions for (a) the static torsional rigidity, (b) the asymptotic group (and phase) velocity at zero frequency and infinite wavelength, (c) the warping cut-off frequency and (d) the contour-shear cut-off frequency.

From equations (28), (44), (36) and (37) the approximate theory yields for these quantities

$$C = \frac{4\mu I^*_{02}I^*_{20}}{I^*_{02}+I^*_{20}}, \tag{45}$$

$$v_g^2 = 4\frac{\mu}{\rho}\frac{I^*_{02}I^*_{20}}{(I^*_{02}+I^*_{20})^2}, \tag{46}$$

$$\omega_w^2 = \frac{\mu}{\rho}\frac{I^*_{02}+I^*_{20}}{I^*_{22}}, \tag{47}$$

$$\omega_s^2 = \frac{\mu}{\rho}\frac{I^*_{00}(I^*_{02}+I^*_{20})}{I^*_{02}I^*_{20}}. \tag{48}$$

Once values of C, v_g, ω_w and ω_s have been determined from the three-dimensional theory for a particular material and cross section, substituting these values into the left hand sides of equations (45)–(48) provides four equations for the four parameters $\kappa_{00}, \kappa_{02}, \kappa_{20}, \kappa_{22}$.

The formulation of the problem of static torsion within the framework of the three-dimensional theory of elasticity and its solution for a large number of cross sectional geometries is due to Saint-Venant [32, 29, 28, 1]. Values of C for a variety of cross sectional shapes are available in the literature and numerical and experimental techniques have been developed for handling situations in which C is difficult to obtain analytically. Expositions of Saint-Venant's treatment of the torsion problem and references to the known solutions can be found in many textbooks on elasticity (see, e.g. [30, 31 and 33]).

Before moving on to a discussion of v_g, ω_w and ω_s it is interesting to note Saint-Venant's researches on the use of a formula of the same form as equation (45) as an approximation to the torsional rigidity.

The first real progress towards a theory of torsion for rods with noncircular cross sections was in a paper by Cauchy [13] in which (following Poisson) an expansion procedure somewhat similar to the one employed in the present paper was used to investigate the torsional vibrations of rods with rectangular cross sections. In Cauchy's paper a quantity analogous to $u_3^{(1,1)}$ (i.e. a warping displacement) was introduced, however Cauchy also introduced the assumption that (in the present notation) $u_2^{(1,0)} = -u_1^{(0,1)}$ and thus limited himself (as we shall show in Section 10) to a very low frequency, very long wavelength approximation to the present theory. Cauchy's theory was rederived by Saint-Venant [14] and extended to apply to cross sections of other geometries. For the torsional rigidity Saint-Venant gave the formula (again using the present notation)

$$C = 4\mu \frac{I_{02} I_{20}}{I_{02}+I_{20}} \tag{49}$$

where I_{02} and I_{20} were the principal moments of inertia of the cross section.

The transition from constructing approximate theories of torsion to obtaining the corresponding exact solutions of the three-dimensional theory of elasticity came in a series of papers by Saint-Venant [32, 29, 28] which preceded his famous memoir on torsion. In these papers Saint-Venant correctly formulated and solved the static torsion problem for the rectangular [29] and elliptical [28] cross sections. For the rectangular section the result for the torsional rigidity was quite complicated, however for the ellipse, Saint-Venant showed that the torsional rigidity given by (49) was exact.† With these exact solutions, Saint-Venant proceeded to discuss the results of his and Cauchy's approximate theory. He concluded that Cauchy's formula for the torsional rigidity of a rectangle [which can be obtained by inserting the value $I_{02} = 4ab^3/3$, $I_{20} = 4a^3b/3$, appropriate to a rectangle of sides $2a$, $2b$, into formula (49)] was incorrect except in the limiting case $a/b \to \infty$, however he noted that a formula of the form of (49) with a multiplicative correction factor could be made exact. He pursues this point in his memoir [1] and actually gives values for this correction factor for various width-to-thickness ratios of a rectangular rod.

In this and later work (see, e.g. [34]), on the subject of torsion, Saint-Venant again considers the possibility of using the exact solution for an elliptical cross section as an approximation for sections of arbitrary cross section, but he seems to prefer the form

$$C = \frac{\mu}{4\pi^2} \frac{(I_{00})^4}{(I_{02}+I_{20})} \tag{50}$$

to which (49) can be transformed in the case of an ellipse. In this form only the area (I_{00}) and the polar moment of inertia ($I_{02}+I_{20}$) of the cross section are involved. In his paper of 1879 [34], Saint-Venant shows that (50) is a good approximation for the torsional rigidity for all but a small number of the cross sectional shapes which he has treated exactly, and he uses (49) only for the ellipse. In view of the assumption we have made concerning the

† For an ellipse with semimajor and semiminor axes a and b

$$I_{00} = \pi ab, \qquad I_{02} = \pi ab^3/4, \qquad I_{20} = \pi a^3 b/4, \qquad I_{22} = \pi a^3 b^3/24,$$

and equation (49) can be written as $C = \mu\pi a^3 b^3/(a^3+b^2)$, which is the form most often reproduced in modern textbooks.

symmetry properties of the cross section it may be that Saint-Venant was expecting too much from the formula (49) when he applied it to triangles, sectors of circles and other sections lacking two perpendicular planes of symmetry.

Moving on to exact solutions of dynamical torsion problems, to the authors' knowledge the only cross sectional shape for which the asymptotic (at low frequency and infinite wavelength) group velocity v_g has been calculated from the three-dimensional theory is the circular cylinder to which we have already referred. However (following Saint Venant [15]), we conjecture that the formula $v_g^2 = C/\rho J$ where C is the static torsional rigidity and J the polar moment of inertia of the cross section is the appropriate expression for the asymptotic group velocity of the three-dimensional theory.

For an isotropic elastic rod the problem of determining the warping cut-off frequency, ω_w, reduces to finding the appropriate solution of the boundary-value problem

$$\mu\nabla^2 u_3 = \rho\ddot{u}_3 \qquad \text{in } R,$$
$$\frac{\partial u_3}{\partial n} = 0 \qquad \text{on } B, \tag{51}$$

where u_3 is the displacement in the axial (x_3) direction, ∇^2 is the two-dimensional Laplacian operator, $\nabla^2 \equiv \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, n is the unit outward normal to the bounding curve, B, of the region, R, occupied by the cross section, and the displacements u_1 and u_2 are identically zero. A discussion of this mathematical problem (which arose in a similar connection) and some of its known solutions has been given by Mindlin and Deresiewicz [35] (see also [36]). In their paper they note that an identical mathematical problem arises in considering the small oscillations of a fluid in a basin and the small vibrations of a gas in a rigid cylindrical container. The only difference between the problem investigated by Mindlin and Deresiewicz and the present problem is the symmetry of the desired solution. In the present case we are interested in the solution that is antisymmetric with respect to the two lines of symmetry in the cross section.

The determination of the lowest contour-shear cut-off frequency, ω_s, within the framework of the three-dimensional theory involves the solution of a particular plane strain vibration problem which does not seem to have been treated in the literature. The mathematically identical problem of the contour modes of a thin plate described by the equations of generalized plane stress has, however, been investigated for a few geometries (see, e.g. [4 pp. 497–498, 37–39] and the references therein) and it is only necessary to reinterpret the elastic constants appearing in solutions of the latter problem to obtain solutions of the former problem.

Example: circular cylinder

As a simple though important example of the determination of correction factors we consider the case of a circular cylindrical rod. From the exact Pochhammer solution [2] we find $C = \mu J = \mu\pi a^4/2$ and $v_g^2 = \mu/\rho$ where a is the radius of the cylinder. Furthermore, the boundary-value problem for determining the warping cut-off frequency, ω_w, reduces to finding the lowest root ($\omega = \omega_w$) of the equation

$$J_2'[\omega a(\rho/\mu)^{\frac{1}{2}}] = 0, \tag{52}$$

where J_2 is the Bessel function of order two and the prime denotes differentiation with respect to its argument. We find

$$\omega_w = \frac{3\cdot0542}{a}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}. \tag{53}$$

The boundary-value problem for finding the contour–shear cut-off frequency reduces to finding the lowest root ($\omega = \omega_s$) of the equation

$$[\psi_2(Ka)-2-\gamma^2/3][\psi_2(K'a)-2-\gamma^2/3]-4[1-\gamma^2/3]^2 = 0, \tag{54}$$

where

$$\begin{aligned} K &= \omega/v_1, \qquad K' = \omega/v_2, \qquad \gamma^2 = (K')^2a^2/2, \\ v_1^2 &= (\lambda+2\mu)/\rho, \qquad v_2^2 = \mu/\rho, \end{aligned} \tag{55}$$

and $\psi_2(x) \equiv xJ_1(x)/J_2(x)$ is Onoe's function [40] of the first kind and order two. Solutions of equation (54) depend on the value of Poisson's ratio ν. For $\nu = 0\cdot3$ we find

$$\omega_s = \frac{2\cdot3479}{a}\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}. \tag{56}$$

Noting that for a circle $I_{02} = I_{20} = \pi a^4/4$, $I_{00} = \pi a^2$ and $I_{22} = \pi a^6/24$, we find from equations (45)–(48) that (for $\nu = 0\cdot3$)

$$\kappa_{02} = \kappa_{20} = 1, \qquad \kappa_{22} = 1\cdot2864, \qquad \kappa_{00} = 0\cdot68908. \tag{57}$$

With these values of the correction factors we return to the problem of the propagation of plane waves formulated in Section 6 for a general cross section and we now specialize to the case of a circular cylinder.

9. PLANE WAVES IN AN INFINITE CIRCULAR CYLINDER

Some of the main features of the dispersion relation (39) of Section 7 can be brought out by considering the simple case of a circular cylindrical rod. Specializing the equations of Sections 5, 6 and 7 to the case of a circular cylinder we find that equations (29) become

$$\begin{aligned} J_+ &= \pi a^4/2, \qquad J_- = 0, \qquad \delta = 0, \\ r^2 &= a^2/2\kappa_{00}, \qquad r_s^2 = a^2/8\kappa_{00}, \qquad r_w^2 = \kappa_{22}a^2/12. \end{aligned} \tag{58}$$

Also, equations (30) reduce to

$$\psi_1 = \tfrac{1}{2}(u_2^{(1,0)}+u_1^{(0,1)}), \qquad \psi_2 = \tfrac{1}{2}(u_2^{(1,0)}-u_1^{(0,1)}), \qquad \psi_3 = r_s u_3^{(1,1)}, \tag{59}$$

equations (40) reduce to

$$\begin{aligned} a_{11} &= \Omega^2-1-\phi^2, \\ a_{22} &= \Omega^2-\phi^2, \\ a_{33} &= \Omega^2(r_w/r_s)^2-1-2(1+\nu)(r_w/r_s)^2\phi^2, \\ a_{13} &= -\phi, \\ a_{23} &= 0, \end{aligned} \tag{60}$$

and the dispersion relation (39) can be written in the simple form

$$a_{22}(a_{11}a_{33}-a_{13}^2) = 0, \tag{61}$$

which gives the two independent dispersion relations

$$a_{22} = 0, \qquad a_{11}a_{33}-a_{13}^2 = 0. \tag{62}$$

The first of these gives the branch

$$\Omega^2 = \phi^2 \tag{63}$$

which is identical to the nondispersive torsional branch of the exact theory. The remaining dispersion relation can be written as

$$(\Omega^2-1-\phi^2)[\Omega^2(r_w/r_s)^2-1-2(1+\nu)(r_w/r_s)^2\phi^2]-\phi^2 = 0, \tag{64}$$

which is quadratic in Ω^2 and ϕ^2. At $\phi = 0$, equation (64) factors into the product of two terms, viz.

$$(\Omega^2-1)[\Omega^2(r_w/r_s)^2-1] = 0, \tag{65}$$

the solutions of which are the contour–shear and warping cut-off frequencies respectively. For $\phi \neq 0$ equation (64) yields for each real, positive value of Ω two values of ϕ^2 which may be real or complex. These two values of ϕ^2 define what we shall call the contour–shear and warping branches of the dispersion relation. The positive torsional, contour–shear and warping branches are sketched in Fig. 2. Of special interest is the fact that the contour–shear branch has a minimum in the Ω, *Re* ϕ plane from which emanate two complex conjugate branches. These complex branches are the continuation of the contour–shear and warping branches down to zero frequency. The intersections of these branches with the $\Omega = 0$ plane are obtained from (64) by setting $\Omega = 0$, with the result

$$\phi^2 = -\frac{1}{2}\pm\frac{1}{2}\left[1-\frac{3}{\kappa_{00}\kappa_{22}(1+\nu)}\right]^{\frac{1}{2}}. \tag{66}$$

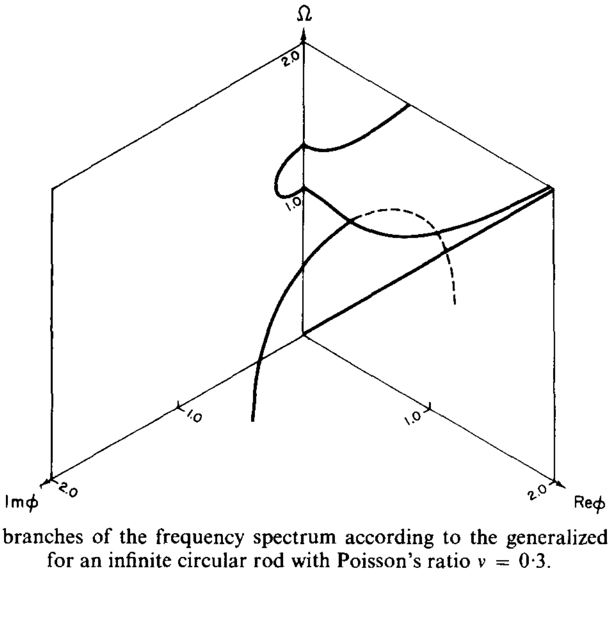

FIG. 2. The three branches of the frequency spectrum according to the generalized torsional theory for an infinite circular rod with Poisson's ratio $\nu = 0{\cdot}3$.

For $0 \leq \nu \leq \frac{1}{2}$ the quantity $3/[\kappa_{00}\kappa_{22}(1+\nu)] > 1$ which means that the wavenumbers given by (66) are always complex. These complex wavenumbers have significance in the solution of problems involving finite cylinders where they correspond to displacements confined near the ends of the cylinder.

The dispersion relation (64) (see also Fig. 2) is qualitatively (and at the cut-off frequencies, quantitatively) the same as the two lowest so-called "flexural branches of circumferential order two" (see, e.g. [41]) of the exact three-dimensional theory. We shall give a detailed comparison of the present approximate dispersion relation and the exact dispersion relation in a forthcoming paper.

The major simplification in the dispersion relation which leads to the uncoupling of the torsional branch from the contour–shear and warping branches is the identity of I_{02}^* to I_{20}^* (hence $J = 0$, $\delta = 0$) for the circular cylinder. In this case the torsional branch is nondispersive in agreement with the exact theory. Clearly, the same simplification (and the nondispersive torsional branch) will occur whenever $I_{02} = I_{20}$ (as for example in a square cross-section), unless $\kappa_{02} \neq \kappa_{20}$†. For other cross sections such as ellipses and rectangles preliminary calculations show that even without correction factors the lowest torsional branch couples with the contour–shear and warping branches and becomes dispersive.

10. REDUCTION TO THE CAUCHY–SAINT-VENANT THEORY

At frequencies low in comparison to the contour–shear and warping cut-off frequencies and at wavelengths long in comparison to a characteristic dimension of the cross section the warping stress-moment gradient $T_{33,3}^{(1,1)}$ and the warping inertia $\rho I_{22}^* \ddot{u}_3^{(1,1)}$ can be neglected in the third of equations (21) with the result

$$T_{31}^{(0,1)} + T_{32}^{(1,0)} = 0. \tag{67}$$

Inserting the second and third of equations (22) into (67) and solving for $u_3^{(1,1)}$ we find

$$u_3^{(1,1)} = -\frac{(I_{02}^* u_{1,3}^{(0,1)} + I_{20}^* u_{2,3}^{(1,0)})}{I_{02}^* + I_{20}^*}, \tag{68}$$

and using this result in equations (22) there results

$$T_{32}^{(1,0)} = -T_{31}^{(0,1)} = \mu \frac{I_{02}^* I_{20}^*}{I_{02}^* + I_{20}^*} (u_2^{(1,0)} - u_1^{(0,1)})_{,3}. \tag{69}$$

Adding the first and second of equations (21) and employing (67) we obtain

$$-2T_{12}^{(0)} = \rho[I_{02}^* \ddot{u}_1^{(0,1)} + I_{20}^* \ddot{u}_2^{(1,0)}], \tag{70}$$

which, neglecting the inertia terms at low frequencies, requires $T_{12}^{(0)} = 0$. The first of equations (22) reduces to

$$u_2^{(1,0)} = -u_1^{(0,1)} \tag{71}$$

† If $I_{02}^* = I_{20}^*$ equation (25) predicts that no warping will occur in the problem of static torsion. For a square cross section $I_{02} = I_{20}$, and it is known from the exact solution that warping does occur. Therefore for the present theory to give the correct qualitative behavior for a square cross section it is necessary that $\kappa_{02} \neq \kappa_{20}$.

a *result* which has traditionally been taken as a starting point for approximate theories. Subtracting the first of equations (21) from the second and employing (69) and (71) we find

$$(C^*\theta_{,3})_{,3} = \rho J^*\ddot{\theta}, \tag{72}$$

where

$$\theta \equiv \tfrac{1}{2}(u_2^{(1,0)} - u_1^{(0,1)}) \tag{73}$$

is the angle of twist and

$$C^* \equiv \frac{4\mu I_{02}^* I_{20}^*}{I_{02}^* + I_{20}^*}, \qquad J^* \equiv I_{02}^* + I_{20}^*. \tag{74}$$

Equations (72)–(74) with the correction factors chosen according to the procedure outlined in Section 8 are equivalent to the classical Saint-Venant theory discussed in the introduction. Without correction factors (i.e. $\kappa_{02} = \kappa_{20} = 1$) equations (72)–(74) are equivalent to an earlier approximate theory due to Cauchy [13] and Saint-Venant [15].

Acknowledgement—This work was supported in part by the National Science Foundation under a grant to Yale University.

REFERENCES

[1] B. de Saint-Venant, Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exercant simultanément. *Mém. prés. div. Sav. Acad. Sci. Inst. Fr.* **14**, 233–560 (1856).
[2] L. Pochhammer, Ueber die Fortpflanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder. *J. reine angew. Math.* **81**, 324–336 (1876).
[3] C. Chree, The equations of an isotropic elastic solid in polar and cylindrical coordinates, their solution and application. *Trans. Camb. phil. Soc.* **14**, 250–369 (1889).
[4] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, pp. 287–289, 4th edition. Cambridge University Press (1927); also Dover (1944).
[5] G. J. Kynch, The fundamental modes of vibration of uniform beams for medium wavelengths. *Br. J. appl. Phys.* **8**, 64–73 (1957).
[6] G. J. Kynch and W. A. Green, Vibrations of beams—I. Longitudinal modes. *Q. Jl Mech. appl. Math.* **10**, 63–73 (1957).
[7] W. A. Green, Vibrations of beams—II. Torsional modes. *Q. Jl Mech. appl. Math.* **10**, 74–78 (1957).
[8] W. A. Green, Vibrations of beams—III. Screw modes. *Q. Jl Mech. appl. Math.* **12**, 22–28 (1959).
[9] N. J. Nigro, Steady-state wave propagation in infinite bars of noncircular cross section. *J. acoust. Soc. Am.* **40**, 1501–1508 (1966).
[10] W. B. Frazer, Stress wave propagation in rectangular bars. *Int. J. Solids Struct.* **5**, 379–397 (1969).
[11] W. A. Green, Dispersion relations for elastic waves in bars. *Progress in Solid Mechanics*, Vol. 1, pp. 225–261. North-Holland (1960).
[12] S. D. Poisson, Mémoire sur l'équilibre et le mouvement des corps élastiques. *Mem. Acad. Sci. Inst. Fr.* **8**, 357–570 (1829).
[13] A. L. Cauchy, Sur la torsion et les vibrations tournantes d'une verge rectangulaire. *Exercises de Mathématiques*, Vol. 4, pp. 47–64. Bure Frères (1829).
[14] B. de Saint-Venant, Mémoire sur la torsion des prismes à base rectangle et à base losange, et sur une petite correction numérique à faire subir, en général, aux moments de torsion. *C. r. hebd. Séanc. Acad. Sci. Paris* **17**, 1180–1190 (1843).
[15] B. de Saint-Venant, Mémoire sur les vibrations tournantes des verges élastiques. *C. r. hebd. Séanc. Acad. Sci. Paris* **28**, 69–72 (1849).
[16] E. Volterra, Second approximation of method of internal constraints and its applications. *Int. J. mech. Sci.* **3**, 47–67 (1961).

[17] A. D. S. BARR, Torsional waves in uniform rods of non-circular section. *J. mech. Engng Sci.* **4**, 127–135 (1962).
[18] W. H. WARNER, The dynamical equations for beams. *Developments in Mechanics*, Vol. 3. Part 2. *Dynamics and Fluid Mechanics*, pp. 119–130. Wiley (1965).
[19] S. K. SHEAR and A. B. FOCKE, The dispersion of supersonic waves in cylindrical rods of polycrystalline silver, nickel and magnesium. *Phys. Rev.* **57**, 532–537 (1940).
[20] S. SPINNER and R. C. VALORE, Comparison of theoretical and empirical relations between the shear modulus and torsional resonance frequencies for bars of rectangular cross section. *J. Res. natn. Bur. Stand.* **60**, 459–464 (1958).
[21] G. J. KYNCH, Longitudinal and screw vibrations of beams. *Nature* **175**, 559 (1955).
[22] R. D. MINDLIN, High frequency vibrations of plated, crystal plates. *Progress in Applied Mechanics; The Prager Anniversary Volume*, pp. 73–84. Macmillan (1963).
[23] R. D. MINDLIN and M. A. MEDICK, Extensional vibrations of elastic plates. *J. appl. Mech.* **26**, 561–569 (1959).
[24] R. D. MINDLIN and G. HERRMANN, A one-dimensional theory of compressional waves in an elastic rod. *Proc. 1st U.S. natn. Congr. Appl. Mech.* pp. 187–191. A.S.M.E. (1952).
[25] R. D. MINDLIN and H. D. MCNIVEN, Axially symmetric waves in elastic rods. *J. appl. Mech.* **27**, 145–151 (1960).
[26] M. A. MEDICK, One-dimensional theories of wave propagation and vibrations in elastic bars of rectangular cross section. *J. appl. Mech.* **33**, 489–495 (1966).
[27] M. A. MEDICK, On the dispersion of longitudinal waves in rectangular bars. *J. appl. Mech.* **34**, 714–717 (1967).
[28] B. DE SAINT-VENANT, Suite au mémoire sur la torsion des prismes. *C. r. hebd. Séanc. Acad. Sci. Paris* **24**, 847–849 (1847).
[29] B. DE SAINT-VENANT, Mémoire sur la torsion des prismes et sur la forme affectée par leurs sections transversales primitivement planes. *C. r. hebd. Séanc. Acad. Sci. Paris* **24**, pp. 485-488 (1847).
[30] S. TIMOSHENKO and J. N. GOODIER, *Theory of Elasticity*, pp. 258–313, 2nd edition. McGraw-Hill (1951).
[31] I. S. SOKOLNIKOFF, *Mathematical Theory of Elasticity*, pp. 107–197, 2nd edition. McGraw-Hill (1956).
[32] B. DE SAINT-VENANT, Mémoire sur l'équilibre des corps solides, dans les limites de leur élasticité, et sur les conditions de leur résistance, quand les déplacements éprouvés par leurs points ne sont pas très-petits. *C. r. hebd. Séanc. Acad. Sci. Paris* **24**, 260–263 (1847).
[33] V. V. NOVOZHILOV, *Theory of Elasticity* (Translated by J. J. SHORR-KON), pp. 213–243. Israel Program for Scientific Translations, Jerusalem (1961).
[34] B. DE SAINT-VENANT, Sur une formule donnant approximativement le moment de torsion. *C. r. hebd. Séanc. Acad. Sci. Paris* **88**, 142–147 (1879).
[35] R. D. MINDLIN and H. DERESIEWICZ, Timoshenko's shear coefficient for flexural vibrations of beams. *Proc. 2nd U.S. natn. Congr. Appl. Mech.* pp. 175–178. A.S.M.E. (1955).
[36] J. H. BALTRUKONIS and M. CHI, Collatz bounds and the collocation method. *Developments in Mechanics*, Vol. 3, Part 2. *Dynamics and Fluid Mechanics*, pp. 147–154. Wiley (1965).
[37] M. ONOE, Contour vibrations of isotropic circular plates. *J. acoust. Soc. Am.* **28**, 1158–1162 (1956).
[38] M. ONOE, Gravest contour vibration of thin anisotropic circular plates. *J. acoust. Soc. Am.* **30**, 634–638 (1958).
[39] R. HOLLAND, Numerical studies of elastic-disc contour modes lacking axial symmetry. *J. acoust. Soc. Am.* **40**, 1051–1057 (1966).
[40] M. ONOE, *Tables of Modified Quotients of Bessel Functions of the First Kind for Real and Imaginary Arguments*. Columbia University Press (1958).
[41] T. R. MEEKER and A. H. MEITZLER, Guided wave propagation in elongated cylinders and plates. *Physical Acoustics*, Vol. 1, Part A, pp. 139–141. Academic Press (1964).

(*Received* 11 *June* 1969)

Абстракт—Выводится приближенная динамическая теория кручения заключающая эффекты депланации и плоской деформации сдвига, которые, вобще сопуствуют деформации кручения в цилиндрических стержнях.